NORTH-HOLLAND

The Combinatorial Power of the Companion Matrix

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ABSTRACT

Using combinatorial methods, we obtain the explicit polynomials for all elements in an arbitrary power of the companion matrix depending on n variables and provide some interesting applications and relationships to Waring's formula on symmetric functions, the general solution to homogeneous linear recurrence relations, the multiplicative inverse of formal power series, the generating function of compositions (of numbers), a unified approach to Chebyshev polynomials including two recently discovered classes that satisfy analogous smallest-norm and orthogonality properties subject to different weight functions, Dickson polynomials of various kinds arising from the theory of finite fields, combinatorial expansions of Toeplitz matrices, and the recent notion of cycle dissections involving a bijective study of Waring's formula.

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1. INTRODUCTION

This paper is motivated by a combinatorial matrix treatment of Waring's formula in connection with Chebyshev and Dickson polynomials. We obtain an explicit formula for the elements in the nth power of the companion matrix and give some interesting applications.

2. THE COMPANION MATRIX

Throughout this paper we are concerned with the companion matrix [17, 19]:

$$C_m = \begin{pmatrix} u_1 & u_2 & u_3 & \cdots & u_{m-1} & u_m \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{2.1}$$

where $u_1, u_2, \ldots, u_m$ may be regarded as indeterminates. For clarity, we may use the notation $C_m(u_1, u_2, \ldots, u_m)$ to denote the matrix (2.1).

This matrix has frequent occurrences in linear algebra, and it has the following characteristic equation:

$$\lambda^m = u_1 \lambda^{m-1} + u_2 \lambda^{m-2} + \cdots + u_m. \tag{2.2}$$

By the Cayley-Hamilton theorem, C_m satisfies the same equation:

$$C_m^m = u_1 C_m^{m-1} + u_2 C_m^{m-2} + \cdots + u_m I_m. \tag{2.3}$$

3. A COMBINATORIAL TREATMENT

The main result of this paper is the following explicit formula for all elements in the nth power of the companion matrix. The proof of this theorem is based on the idea of using a digraph to represent a matrix.

THEOREM 3.1. *The (i, j) entry $c_{ij}^{(n)}(u_1, \ldots, u_m)$ in the matrix $C_m^n(u_1, u_2, \ldots, u_m)$ is given by the following formula:*

$$c_{ij}^{(n)}(u_1, \ldots, u_m) = \sum_{(k_1, k_2, \ldots, k_m)} \frac{k_j + k_{j+1} + \cdots + k_m}{k_1 + k_2 + \cdots + k_m} \times \binom{k_1 + \cdots + k_m}{k_1, \ldots, k_m} u_1^{k_1} \cdots u_m^{k_m}, \qquad (3.1)$$

where the summation is over nonnegative integers satisfying $k_1 + 2k_2 + \cdots + mk_m = n - i + j$, and the coefficient in (3.1) is defined to be 1 if $n = i - j$.

Proof. It is known that the powers of a matrix can be explained in terms of paths in a directed graph corresponding to the matrix [3]. For the matrix C_m, we define the digraph D as a weighted digraph with vertex set $\{v_1, v_2, \ldots, v_m\}$, and with arcs $(v_1, v_1), (v_1, v_2), (v_1, v_3), \ldots, (v_1, v_m)$ and $(v_2, v_1), (v_3, v_2), (v_4, v_3), \ldots, (v_m, v_{m-1})$ in which (v_1, v_i) has weight u_i and (v_i, v_{i-1}) has weight 1, as shown in Figure 1.

Observe that the outdegree of any vertex in D is one, except for the vertex v_1, which is adjacent to any vertex. The entry $c_{ij}^{(n)}$ is determined by the paths of length n from v_i to v_j in the digraph D. Clearly, when $n - i + j$ is negative, there is no path of length n from v_i to v_j. When $n = i - j$, there is a unique path from v_i to v_j of weight 1 and length $i - j$. We may now assume that $n - i + j > 0$. Let us analyze the paths from v_i to v_j according to the following cases: $i = j$, $i < j$, $i > j$.

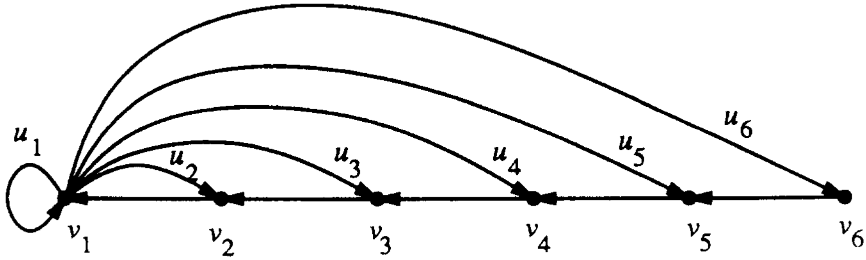

FIG. 1. The digraph for C_6.

The most fundamental concern is the structure of paths from v_1 to itself. Since the outdegree of any vertex v_i is one for $i \geqslant 2$, we may denote the unique path from v_i to v_1 by $v_i \Rightarrow v_1$. Consequently, a path P of length s from v_1 to v_1 is always of the form

$$v_1 \to v_{\lambda_1} \Rightarrow v_1 \to v_{\lambda_2} \Rightarrow v_1 \to v_{\lambda_3} \Rightarrow v_1 \to \cdots \Rightarrow v_1 \to v_{\lambda_r} \Rightarrow v_1,$$

where

$$s = \lambda_1 + \lambda_2 + \cdots + \lambda_r.$$

Moreover, it is easily seen that the sequence $(\lambda_1, \lambda_2, \dots, \lambda_r)$ is in one-to-one correspondence to a path from v_1 to v_1. In the language of enumerative combinatorics, the sequence $(\lambda_1, \lambda_2, \dots, \lambda_r)$ is called a composition or an ordered partition of s. A composition is said to be of type $1^{k_1}2^{k_2} \cdots m^{k_m}$ if there are k_i occurrences of i. It is easy to see that the number of compositions of type $1^{k_1}2^{k_2} \cdots m^{k_m}$ equals the multinomial coefficient [15, 30]

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \dots, k_m}. \tag{3.2}$$

The above formula coincides with (3.1) for $i = j = 1$. Let us consider the case $i = j$. A path P of length n from v_i to itself must be of the following form:

$$P\colon \quad v_i \Rightarrow v_1 \to \cdots \to v_1 \to v_r \Rightarrow v_i \tag{3.3}$$

for some $r \geqslant i$. Since the last segment $v_r \Rightarrow v_i$ in the path P plays a particular role, the path P is essentially determined by a path Q of length $n - r$ from v_1 to v_1 for some $r \geqslant i$. Given a type $1^{k_1}2^{k_2} \cdots m^{k_m}$, the part r can be chosen from those parts for which $k_r > 0$. Moreover, with r and the type given, the number of paths from v_i to v_i of weight $u_1^{k_1}u_2^{k_2} \cdots u_m^{k_m}$ equals

$$\binom{k_1 + k_2 + \cdots + k_m - 1}{k_1, k_2, \dots, k_r - 1, \dots, k_m} = \frac{k_r}{k_1 + k_2 + \cdots + k_m}\binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \dots, k_m}. \tag{3.4}$$

Because the above identity holds even for $k_r = 0$, it follows that the formula (3.1) is valid for $i = j$.

Let us proceed to consider the case $i < j$. Any path of length n from v_i to v_j must start with a path from v_i to v_1, continue with a path from v_1 to v_1 followed by an arc (v_1, v_r) for some $r \geqslant j$, and end with a path from v_r to v_j. Given r and a type $1^{k_1}2^{k_2} \cdots m^{k_m}$, the number of paths from v_i to v_j of weight $u_1^{k_1}u_2^{k_2} \cdots u_m^{k_m}$ equals the same number as in (3.4). Bear in mind that here the type $1^{k_1}2^{k_2} \cdots m^{k_m}$ corresponds to the number $n - i + j$. One may also treat this case by adding the path from v_j to v_i to form a path from v_i to v_i, so that the previous argument applies.

Finally, we treat the case $i > j$. Any path from v_i to v_j starts with a path from v_i to v_j, then the rest is a path from v_j to v_j of length $n - i + j$. Hence the argument for the case $i = j$ takes care of the rest of the proof. ∎

We note that the above theorem can be easily extended to the infinite companion matrix $C_m(u_1, u_2, \ldots)$.

As suggested by the referee, we give the generating function for $c_{ij}^{(n)}(u_1, u_2, \ldots, u_m)$, which can be derived by the same combinatorial argument used to prove Theorem 3.1, or by applying the formula for the inverse of a matrix to $(I - tC_m)^{-1}$.

THEOREM 3.2. *We have the following generating functions*:

$$\sum_{n=0}^{\infty} c_{ij}^{(n)}(u_1, u_2, \ldots, u_m)t^n$$

$$= \begin{cases} \dfrac{t^i u_j + t^{i+1}u_{j+1} + \cdots + t^{m+i-j}u_m}{1 - tu_1 - t^2u_2 - \cdots - t^m u_m} & \textit{if} \quad i < j, \\[2ex] \dfrac{t^{i-j}\left(1 - tu_1 - t^2u_2 - \cdots - t^{j-1}u_{j-1}\right)}{1 - tu_1 - t^2u_2 - \cdots - t^m u_m} & \textit{if} \quad i \geqslant j. \end{cases}$$

The two cases may be combined into the formula

$$\sum_{n=0}^{\infty} c_{ij}^{(n)}(u_1, u_2, \ldots, u_m)t^n = \chi(i \geqslant j)t^{i-j} + \frac{t^i u_j + t^{i+1}u_{j+1} + \cdots + t^{m+i-j}u_m}{1 - tu_1 - t^2u_2 - \cdots - t^m u_m},$$

where $\chi(i \geqslant j)$ is 1 if $i \geqslant j$ and 0 otherwise.

Proof. We only outline the proof for the case $i = j$, because the other cases fall into the same scheme. Let D be the digraph defined in the proof of Theorem 3.1, and tD the digraph obtained from D by multiplying every weight by a factor t. In this way, the power of t carries the length of a path when the weight of a path is calculated. The generating function $c_{ij}^{(n)}$ is the (i,j) entry in the matrix

$$I + tC_m + t^2 C_m + \cdots ,$$

namely, the weighted sum of all paths from v_i to v_j in tD. For the case $i = j$, the path from v_i to v_i of length zero contributes 1 to the generating function. A path P from v_i to v_i of length greater than zero must start with a path from v_i to v_1, whose weight is t^{i-1}, and continue with a path from v_1 to v_1, whose generating function equals

$$\sum_{k=0}^{\infty} \left(tu_1 + t^2 u_2 + \cdots + t^m u_m\right)^k = \frac{1}{1 - tu_1 - t^2 u_2 - \cdots - t^m u_m}.$$

Finally, the path P ends with a path containing the arc (v_1, v_r) for some $r \geqslant i$, and the shortest path from v_r to v_i. The generating function of the last path equals

$$t^i u_i + t^{i+1} u_{i+1} + \cdots + t^m u_m .$$

Summing all segments, one obtains the generating function for $c_{ii}^{(n)}(u_1, u_2, \ldots, u_m)$:

$$1 + \frac{t^i u_i + t^{i+1} u_{i+1} + \cdots + t^m u_m}{1 - tu_1 - t^2 u_2 - \cdots - t^m u_m},$$

which coincides with the desired expression. ∎

The above generating-function formula can be used to derive the generating function for Waring's formula, Dickson polynomials, and Chebyshev polynomials, which will be discussed later.

4. WARING'S FORMULA

Let us consider the companion matrix in the elementary symmetric functions in $x_1, x_2, \ldots, x_m$ [22]; namely, let $u_i = (-1)^{i-1}e_i(x_1, x_2, \ldots, x_m)$, where e_i is the ith elementary symmetric function. Then $x_1, x_2, \ldots, x_m$ are the eigenvalues of the companion matrix $C_m(u_1, u_2, \ldots, u_m)$, and the power sum symmetric function $p_n = x_1^n + x_2^n + \cdots + x_m^n$ equals the trace of C_m^n. Clearly, Theorem 3.1 implies Waring's formula [2, 5, 6, 9, 21, 27]:

COROLLARY 4.1 (Waring's formula). *Let p_n and e_n be the power sum and elementary symmetric functions in $x_1, x_2, \ldots, x_m$. Then we have*

$$p_n = \sum_{(k_1,k_2,\ldots,k_m)} (-1)^{k_2+k_4+\cdots} \frac{n}{k_1+k_2+\cdots+k_m} \times \binom{k_1+k_2+\cdots+k_m}{k_1, k_2, \ldots, k_m} e_1^{k_1} e_2^{k_2} \cdots e_m^{k_m},$$

where the sum is over $k_1 + 2k_2 + \cdots + mk_m = n$.

Using the generating function for $c_{ij}^{(n)}(u_1, u_2, \ldots, u_m)$, one obtains the generating function for the power sum symmetric functions in terms of the elementary symmetric functions:

COROLLARY 4.2.

$$\sum_{n=0}^{\infty} p_n(x_1, x_2, \ldots, x_m)t^n = \frac{\sum_{k=0}^{m} (-1)^k (m-k) e_k t^k}{\sum_{k=0}^{m} (-1)^k e_k t^k},$$

where e_0 equals 1.

5. DICKSON POLYNOMIALS IN SEVERAL VARIABLES

Dickson polynomials are a variation of Chebyshev polynomials, and have proven highly useful in the study of permutation polynomials over a finite field (see a recent monograph by Lidl, Mullen, and Turnwald [18]). They have been generalized to several variables (see also [1, 18]). Here we present the connection between the companion matrix and Dickson polynomials $D_n^{(1)}(x_1, x_2, \ldots, x_m, a)$ as in [18], which we shall simply write as $D_n(x_1, x_2, \ldots, x_m, a)$.

COROLLARY 5.1. *The Dickson polynomials of the first kind are given by*

$$D_n(x_1, x_2, \ldots, x_m, a) = \operatorname{trace} C_{m+1}^n(u_1, u_2, \ldots, u_{m+1}), \tag{5.1}$$

where

$$u_i = (-1)^{i-1} x_i, \qquad i = 1, 2, \ldots, m,$$

$$u_{m+1} = (-1)^m a.$$

The Dickson polynomials of the second kind are given by

$$\begin{aligned} &E_n(x_1, \ldots, x_m, a) \\ &\quad = c_{1,1}^{(n)}(u_1, u_2, \ldots, u_{m+1}) \\ &\quad = \sum_{k_1 + 2k_2 + \cdots + (m+1)k_{m+1} = n} \binom{k_1 + \cdots + k_{m+1}}{k_1, \ldots, k_{m+1}} u_1^{k_1} u_2^{k_2} \cdots u_{m+1}^{k_{m+1}}, \end{aligned} \tag{5.2}$$

where $u_1, u_2, \ldots, u_{m+1}$ are the same as for $D_n(x_1, \ldots, x_m, a)$.

The relation (5.1) agrees with the result for Dickson polynomials of the first kind given by Lidl et al. [18], after making changes in notation and simplifying their expression. It may also be proved directly from the definition of Dickson polynomials of the first kind. The formula (5.2) for Dickson polynomials of the second kind follows from the generating function definition [18, p. 22] and the observation about the generating function of compositions in Section 10. We omit these details.

We note that the generating functions for $D_n(x_1,, \ldots, x_m, a)$ and $E_n(x_1, \ldots, x_m, a)$ as described in [18] also follow from the generating function for $c_{ij}^{(n)}(u_1, u_2, \ldots, u_m)$.

6. THE VANDERMONDE MATRIX

For completeness we mention a connection between the power of the companion matrix and the Vandermonde matrix. Suppose $x_1, x_2, \dots, x_m$ are the eigenvalues of the companion matrix $C_m(u_1, u_2, \dots, u_m)$. Let $V(x_1, x_2, \dots, x_m)$ be the Vandermonde matrix in $x_1, x_2, \dots, x_m$ written in the following form:

$$\begin{pmatrix} x_1^{m-1} & x_2^{m-1} & \cdots & x_m^{m-1} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_m \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{6.1}$$

Then the companion matrix and the Vandermonde matrix are related via the following equation (see [17, p. 69]).

PROPOSITION 6.1.

$$\begin{aligned} &C_m^n(u_1, u_2, \dots, u_m)V(x_1, x_2, \dots, x_m) \\ &\quad = V(x_1, x_2, \dots, x_m)D(x_1^n, x_2^n, \dots, x_m^n), \end{aligned} \tag{6.2}$$

where $D(x_1^n, \dots, x_m^n)$ is the diagonal matrix.

Clearly, when C_m has distinct eigenvalues it can be diagonalized by the Vandermonde matrix. If we restrict the companion matrix to the elementary symmetric functions in indeterminates $x_1, x_2, \dots, x_m$, namely $u_i = (-1)^{i-1}e_i(x_1, x_2, \dots, x_m)$, then the relation (6.2) is identically true in these indeterminates.

7. LINEAR RECURRENCE RELATIONS

The companion matrix can be generally employed to deal with homogeneous linear recurrence relations [10, p. 123] and linear homogeneous differential equations [7]. Let $C_m(u_1, u_2, \dots, u_m)$ be as before. Consider the homogeneous linear recurrence relations for a sequence $(z_0, z_1, z_2, \dots)$:

$$z_n = u_1 z_{n-1} + u_2 z_{n-2} + \cdots + u_m z_{n-m}. \tag{7.1}$$

It follows that

$$\begin{pmatrix} z_m \\ z_{m-1} \\ \vdots \\ z_1 \end{pmatrix} = C_m(u_1, u_2, \ldots, u_m) \begin{pmatrix} z_{m-1} \\ z_{m-2} \\ \vdots \\ z_0 \end{pmatrix}. \tag{7.2}$$

By iteration one obtains

$$\begin{pmatrix} z_{n+m-1} \\ z_{n+m-2} \\ \vdots \\ z_n \end{pmatrix} = C_m^n(u_1, u_2, \ldots, u_m) \begin{pmatrix} z_{m-1} \\ z_{m-2} \\ \vdots \\ z_0 \end{pmatrix}. \tag{7.3}$$

Thus, the power C_m^n determines the solution to the linear recurrence relation (7.1) in terms of the initial set $(z_0, z_1, \ldots, z_{m-1})$. More specifically, the last row of C_m^n alone (or any other row) gives the solution to (7.1).

8. THE FOUR KINDS OF CHEBYSHEV POLYNOMIALS

As noted in Section 7, the power of the companion matrix essentially determines the solution to a linear recurrence relation. In this section, we are concerned with Chebyshev polynomials, which satisfy a second order linear recurrence relation. The Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$ are of fundamental importance in approximation theory:

$$T_n(x) = \frac{1}{2} \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}, \tag{8.1}$$

$$U_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2x)^{n-2k}. \tag{8.2}$$

There are two less well-known classes of polynomials that are closely related to the $T_n(x)$ and $U_n(x)$. They have been shown to enjoy analogous smallest-norm and orthogonality properties to the classical ones [14, 23].

These two classes were called Chebyshev polynomials of the third kind and fourth kind, and were denoted by $V_n(x)$ and $W_n(x)$. They have also been known under various names in the literature, including airfoil polynomials and half-angle shifted polynomials (see [4, 11, 24, 25]). All the four classes are specializations of the Jacobi polynomials. Here we shall give a unified treatment of these four kinds of Chebyshev polynomials in terms of the power of the following companion matrix:

$$Q = \begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}. \tag{8.3}$$

By Theorem 3.1, one may write down the formulas for Q_{ij}^n. It turns out that the element Q_{11}^n is the same as $U_n(x)$. Moreover, all the others are also Chebyshev polynomials of the second kind:

PROPOSITION 8.1. *We have*

$$\begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} U_n(x) & -U_{n-1}(x) \\ U_{n-1}(x) & -U_{n-2}(x) \end{pmatrix}. \tag{8.4}$$

We could not help wondering if (8.4) is really new. At least it is not as widely known as its counterpart for Fibonacci numbers:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{8.5}$$

From identity (8.4) and $\det Q = 1$, it immediately follows that

$$U_{n-1}^2(x) - U_n(x)U_{n-2}(x) = 1. \tag{8.6}$$

Using the recurrence relation below for $U_n(x)$ and this identity, one gets

$$U_n^2(x) - 2xU_n(x)U_{n-1}(x) + U_{n-1}^2(x) = 1. \tag{8.7}$$

In order to present a unified treatment of the four kinds of Chebyshev polynomials, we regard them as defined by the same recurrence relation with different initial values, although they can all be described as trigonometric functions or Jacobi polynomials with different parameters. Table 1 gives a comparative account of the four kinds of Chebyshev polynomials. They all satisfy the same recurrence relation

$$P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x),$$

with initial value $P_0(x) = 1$ and different values for $P_1(x)$ as listed in the table.

Based on the initial values, the relationships of $T_n(x)$, $V_n(x)$, and $W_n(x)$ to $U_n(x)$ reduce to the following single identity:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \tag{8.8}$$

which is just the recurrence relation for $U_n(x)$, and is equivalent to the Pascal triangle:

$$\binom{n-k}{k} = \binom{n-k-1}{k} + \binom{n-k-1}{k-1}.$$

Notice that the four kinds of Chebyshev polynomials can be expressed by the elements in the matrix Q^n as half the trace, the (1, 1) entry, the row sum, and the column sum. Their generating functions also follow from the generating functions for $c_{ij}^{(n)}(u_1, u_2, \ldots, u_m)$. We note that the well-known Fibonacci polynomials and the Lucas polynomials fall into the framework of the companion matrix as special cases. One could also regard the relation (3.1) as giving a generalization of the Fibonacci numbers upon choosing $u_1 = u_2 = \cdots = u_m = 1$.

TABLE 1

	$T_n(x)$	$U_n(x)$	$V_n(x)$	$W_n(x)$
Trigonometric definition, $x = \cos\theta$	$\cos n\theta$	$\dfrac{\sin(n+1)\theta}{\sin\theta}$	$\dfrac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta}$	$\dfrac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}$
Weight function for orthogonality	$\dfrac{1}{\sqrt{1-x^2}}$	$\sqrt{1-x^2}$	$\dfrac{\sqrt{1+x}}{\sqrt{1-x}}$	$\dfrac{\sqrt{1-x}}{\sqrt{1+x}}$
Weight function for smallest norm	1	$\sqrt{1-x^2}$	$\sqrt{1+x}$	$\sqrt{1-x}$
$P_1(x)$	x	$2x$	$2x-1$	$2x+1$
Relationship to $U_n(x)$	$\dfrac{U_n(x) - U_{n-2}(x)}{2}$	—	$U_n(x) - U_{n-1}(x)$	$U_n(x) + U_{n-1}(x)$

OBSERVATIONS. It seems that the relationships of $V_n(x)$ and $W_n(x)$ to $U_n(x)$ have been unnoticed in approximation theory. Interestingly, $V_n(x)$ and $W_n(x)$ turn out to be the polynomials $G_n(2x)$ and $F_n(2x)$ used in finite fields (see [18, p. 32]). Particularly, $W_n(x)$ is closely related to the polynomials $f_n(x)$ in [13], which play a substantial role in the study of optimal normal bases of finite fields [12, 13, 26], and in the study of primitive polynomials over finite fields [28]. Specifically, $W_n(x)$ and $f_n(x)$ are related by $f_n(2x) = W_n(x)$. As a consequence, the polynomials $f_n(x)$ coincide with the polynomials $F_n(x)$ mentioned above. Chebyshev polynomials also have an important application in mathematical physics in [20], although the role of these polynomials was not then recognized.

9. MULTIPLICATIVE INVERSE OF FORMAL POWER SERIES

In this section, we point out that the multiplicative inverse of formal power series is also related to the power of the companion matrix. We remark that a classical determinantal formula for Chebyshev polynomials of the second kind immediately follows from the classical determinantal formula for the multiplicative inverse of a formal power series.

Given a sequence $z_0, z_1, z_2, \dots$ satisfying the linear recurrence relation (7.1) with initial values $z_0 = 1$ and

$$z_k = \sum_{i=1}^{k} u_i z_{k-i}$$

for $k = 1, 2, \dots, m-1$, then the generating function of the sequence is

$$\sum_{n=0}^{\infty} z_n t^n = \frac{1}{1 - u_1 t - u_2 t^2 - \cdots - u_m t^m}. \tag{9.1}$$

Different initial values would only change the numerator of (9.1). Essentially, the solution to the recurrence relation (7.1) depends on the multiplicative inverse of the finite series $1 - u_1 t - u_2 t^2 - \cdots - u_m t^m$. It is worth noting that there is a determinantal formula involving the Toeplitz matrices for the inverse of an infinite power series. Let

$$\sum_{n=0}^{\infty} v_n t^n = \frac{1}{1 - u_1 t - u_2 t^2 - \cdots}.$$

Then we have (see [8, 29])

$$v_n = \begin{vmatrix} u_1 & u_2 & \cdots & u_{n-1} & u_n \\ -1 & u_1 & \cdots & u_{n-2} & u_{n-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & -1 & u_1 & u_2 \\ \mathbf{0} & & & -1 & u_1 \end{vmatrix}. \tag{9.2}$$

Letting $u_i = 0$ for $i > m$, one obtains the following determinantal formula for z_n in (9.1):

$$z_n = v_n(u_1, \ldots, u_m) = \begin{vmatrix} u_1 & u_2 & \cdots & u_m & & & \mathbf{0} \\ -1 & u_1 & \cdots & u_{m-1} & u_m & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & u_m \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & -1 & u_1 & u_2 \\ \mathbf{0} & & & & & -1 & u_1 \end{vmatrix}. \tag{9.3}$$

It is not hard to see that (9.3) also implies (9.2).

The above formula is related to the companion matrix via the (1, 1) entry of the power $C_m^n(u_1, u_2, \ldots, u_m)$, which is denoted by

$$U_n(x_1, x_2, \ldots, x_m) = \sum_{k_1 + 2k_2 + \cdots + mk_m = n} \binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}. \tag{9.4}$$

This polynomial is essentially the Dickson polynomial of the second kind in several variables, and it also gives the multiplicative inverse of a formal power series. Furthermore, it coincides with the determinant (9.3):

PROPOSITION 9.1.

$$U_n(u_1, u_2, \ldots, u_m) = v_n(u_1, u_2, \ldots, u_m). \tag{9.5}$$

The proof of this proposition will be discussed in the next section from two perspectives. For $m = 2$, it reduces to the well-known identity for

Chebyshev polynomials of the second kind [3, 18]:

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k}(-a)^k x^{n-2k} = \begin{vmatrix} x & a & 0 & \cdots & 0 & 0 \\ 1 & x & a & \cdots & 0 & 0 \\ 0 & 1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & a \\ 0 & 0 & 0 & \cdots & 1 & x \end{vmatrix}_{n\times n} . \tag{9.6}$$

This polynomial in x and a is the Dickson polynomial of the second kind $E_n(x, a)$, and the Chebyshev polynomial $U_n(x)$ equals $E_n(x, 1)$. The determinant (9.3) has also been studied in [16] in connection with difference equations.

10. COMPOSITIONS AND TOEPLITZ MATRICES

The last section of this paper deals with the polynomial $U_n(x_1, x_2, \dots, x_m)$ which arises in the powers of the companion matrix. Notice that in the proof of Theorem 3.1 the multinomial coefficient (3.2) counts the number of compositions of type $1^{k_1}2^{k_2} \cdots m^{k_m}$. Hence the definition (9.4) of $U_n(x_1, x_2, \dots, x_m)$ can be rewritten as

$$U_n(x_1, x_2, \dots, x_m) = \sum_{(\lambda_1, \lambda_2, \dots, \lambda_r)} x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_r}, \tag{10.1}$$

where the sum ranges over compositions $(\lambda_1, \lambda_2, \dots, \lambda_r)$ of n in which each part is positive but not greater than m. Clearly, we may assume without loss of generality that U_n has variables $x_1, x_2, \dots, x_n$. To reduce it to m variables $(m < n)$, one only needs to let $x_{m+1} = \cdots = x_n = 0$.

The form (10.1) tells us that the polynomial U_n is in fact the generating function of compositions of n. Based on this fact, one realizes that the polynomial $U_n(u_1, u_2, \dots, u_m)$ is exactly the solution of the recurrence relation (7.1) with initial values $U_0 = 1$ and

$$U_k(u_1, u_2, \dots, u_m) = \sum_{i=1}^{k} u_i U_{k-i}(u_1, u_2, \dots, u_m),$$

for $1 \leqslant k < m$, and for $n \geqslant m$, we have

$$U_n(u_1, u_2, \dots, u_m) = \sum_{i=1}^{m} u_i U_{n-i}(u_1, u_2, \dots, u_m).$$

Note that the above relation is equivalent to the statement that $(1, U_1, U_2, \ldots)$ is the multiplicative inverse of the sequence $(1, -u_1, -u_2, \ldots)$. Hence one obtains the identity (9.5).

We conclude this paper with a combinatorial expansion of Toeplitz determinants and an analogous expansion of the determinant involving Waring's formula.

The Expansion in Terms of Compositions

There is a one-to-one correspondence between the set of compositions of n and the set of nonzero terms in the expansion of the determinant (9.2). The key observation is that in order to pick up a nonzero term in the expansion of (9.2), if u_k is chosen from the first row, then one has to choose the -1's from the second row to the kth row. After that, the argument goes recursively for the remaining submatrix. Once this fact is noticed, the computation of the determinant (in the sense of expansion) is just a different way of representing the generating function of compositions.

The Expansion in Terms of Cyclic Compositions

In recent work of Chen, Lih, and Yeh [6] on a bijective explanation of Waring's formula, the idea of cyclic tableaux plays a substantial role. Here we make a connection between cyclic tableaux and the following determinantal expression of Waring' formula [22]:

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & 0 & \cdots & 0 \\ 2e_2 & e_1 & 1 & 0 & \cdots & 0 \\ 3e_3 & e_2 & e_1 & 1 & \cdots & 0 \\ 4e_4 & e_3 & e_2 & e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ ne_n & e_{n-1} & e_{n-2} & e_{n-3} & \cdots & e_1 \end{vmatrix} \tag{10.2}$$

By expanding the first column of the above determinant and using the above combinatorial expansion of a Toeplitz determinant, one may see that a nonzero term in the expression of (10.2) corresponds to a composition in which the first part (represented by a row of dots) has a distinguished dot:

$$\circ\ \circ\ \bullet\ \circ \,|\, \circ\ \circ \,|\, \circ \,|\, \circ\ \circ\ \circ\ \circ,$$

which is a composition $(4, 2, 1, 4)$ with a distinguished dot in the first part. This notion of compositions turns out to be equivalent to that of cycle dissections introduced by Chen, Lih, and Yeh [6]. We leave it to the reader to verify or to wonder that by the combinatorial interpretation of the coefficients in the Waring's formula given in [6], the determinantal expression and the polynomial expression of Waring's formula are in one-to-one correspondence with each other.

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